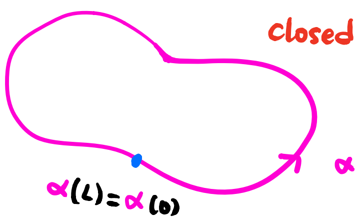
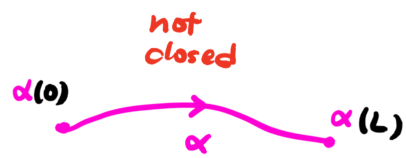
## § Some global theorems for plane curves

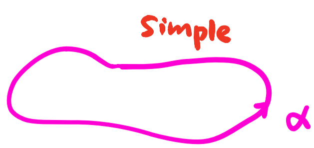
<u>**Def$^{\underline{n}}$**</u>: A plane curve $\alpha : [0, L] \longrightarrow \mathbb{R}^2$ is said to be

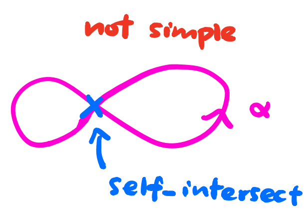
- **closed** if $\alpha^{(k)}(0) = \alpha^{(k)}(L)$ for $k = 0, 1, 2, \ldots$



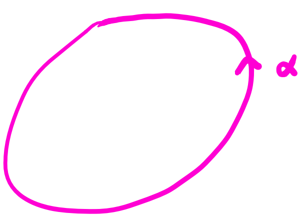




- **Simple** if $\alpha$ is 1-1 except possibly at $s = L$.

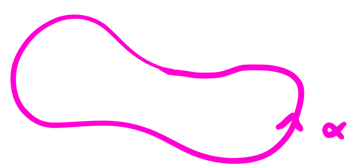



- **Convex** if $k > 0$ everywhere.




Convex

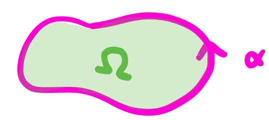



not convex

## (A) Isoperimetric Inequality

For any simple closed curve in $\mathbb{R}^2$,

$$\boxed{A \leq \frac{L^2}{4\pi}}$$




$L$ = Length($\alpha$)

$A$ = Area($\Omega$)

"=" holds <=> round circle

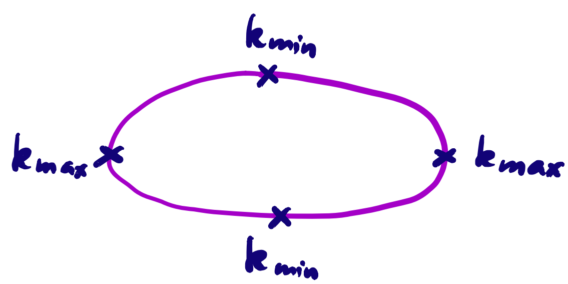
Proof: Exercise.

## (B) Four Vertex Theorem

For any simple closed (convex) curve in $\mathbb{R}^2$,

$\exists$ at least 4 vertices (i.e. points where $k' = 0$)

ellipse has exactly 4 vertices.




Proof: omitted.

Note: It is easy to show that $\exists$ 2 vertices, where $k$ achieves its maximum and minimum. What is non-trivial is that there are at least 2 more!

## (C) Theorem of Turning Tangents

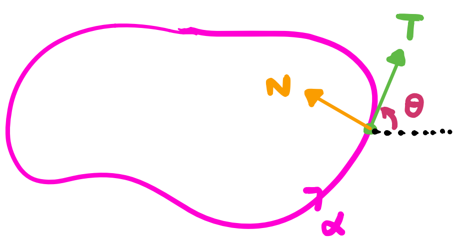
For any simple closed curve $\alpha$ in $\mathbb{R}^2$, which is positively oriented (ie. $N$ points inward)

$$\boxed{\int_\alpha k(s)\,ds = 2\pi}$$

Proof: Let $\theta(s)$ be the angle from the positive x-axis to the unit tangent $T(s) = \alpha'(s)$, where $\alpha$ is p.b.a.l.

$$\boxed{\theta'(s) = k(s)}$$

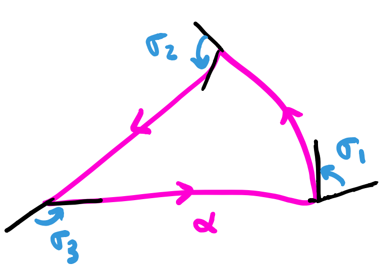
$$\Downarrow$$




$$\underbrace{\theta(L) - \theta(0)}_{=\,2\pi} = \int_0^L k(s)\,ds$$

$\square$

Non-smooth version:




$$\boxed{\int_\alpha k(s)\,ds + \sum_i \sigma_i = 2\pi}$$